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Exercise 2.2.23

a.

W-convergent: $(\forall N \in \mathbb{N})(\exists \varepsilon > 0)(\forall n \geqslant N)[\,|x_n - x| < \varepsilon\,]$

W-convergent sequences require that any chosen N, including $N=1$, have that for $(\forall n \geqslant N)$, $|x_n - x| < \varepsilon$.

This means that $|x_n - x|$ is bounded by some $\varepsilon > 0$, but the sequence does not necessarily converge.

An example is $\sin(n)$, which is bounded by 1.

b. S-convergent: $(\exists N)(\forall \varepsilon > 0)(\forall n \geqslant N)[\,|x_n - x| < \varepsilon\,]$

This means that there must be at least one finite N that satisfies $(\forall \varepsilon > 0)(\forall n \geqslant N)[\,|x_n - x| < \varepsilon\,]$. This condition implies convergence, because $(\forall \varepsilon > 0)$, there exists at least the finite one or more N that S-convergence demands.

Not all convergent sequences are S-convergent, for instance for the sequence $x_n = \frac{1}{n}$ has no finite N for $(\forall \varepsilon > 0)(\forall n \geqslant N)[\,|x_n - x| < \varepsilon\,]$, as $x_{n+1} < x_n$ for this sequence.

c. We first show that W-limits may not be unique:

we do this by counter example:

consider $x_n = \sin(n)$

we know that $|\sin(n)| < 2$, and is hence bounded, and is thus W-convergent, with limit 0.

we know that $\sin(n) - 1 \in [-2, 0]$ and is hence also bounded:

$$|\sin(n) - 1| < 3$$

Thus, this W-convergent sequence has at least the two limits shown, and hence has more than one limit.

d. We now show that S-limits are unique:

We know that all convergent sequences have a unique limit, and we know that all S-convergent sequences are convergent, hence all S-convergent sequences have a unique limit.